Application of Dynamic Programming to Optimizing the Orbital Control Process of a 24-Hour Communication Satellite

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This paper discusses a method for optimizing an orbital control process for a 24-hr communication satellite. The process is concerned with transferring a satellite from its injected orbit to a hypothetical point moving in a nearby specified reference orbit. The reference orbit considered here is a circular, equatorial orbit with a period of one sidereal day. The optimization of this transfer process is accomplished by applying the techniques of dynamic programming. This essentially involves treating the orbit transfer as a multistage decision process and finding the optimal set of velocity vector increments to be applied to the satellite to accomplish the desired transfer subject to a constraint on the mass of fuel used. The numerical results indicate that the linearity assumptions and the iteration technique involved in the computation of the velocity vector increments are entirely satisfactory.

I. Introduction

THIS paper is concerned with the optimization of an orbital control process for a 24-hr communication satellite. It is assumed that the satellite will be the intermediate terminal of a communications link employing narrow-beam fixed antennas at either end. The satellite is to be placed in a circular, equatorial orbit, with a period of one sidereal day and maintained in this orbit within tolerances sufficiently small to enable the ground antennas to observe it at all times over a specified orbital lifetime. Orbital control is the process of placing the satellite in the required orbit within specified tolerances and maintaining it there.

Because of errors in the ascent guidance system, the satellite will be initially injected into an orbit that is slightly elliptic, slightly inclined to the earth's equatorial plane, slightly displaced from its desired position over the equator, and having an incorrect period. The orbital control process to be discussed is concerned with refining this preliminary orbit on the basis of observations made after the final termination of thrust for the ascent phase.

After the desired orbit has been achieved, the orbital control process will be required to correct periodically for the perturbing effects of the gravitational attractions of the sun and moon and distortions in the shape of the earth. These effects produce secular and periodic changes in the two-body parameters which cause the orbital eccentricity and inclination to depart from zero (or near zero). This results in a periodic wandering motion of the satellite within the volume of space common to the cones of view of the terminal antennas. The amplitude of this wandering motion gradually increases due to the gravitation perturbations until, during part of the day, the satellite moves outside of the cones of view of the antennas. In addition, secular effects due to any small error in orbital period cause the periodic wandering motion pattern to drift along the equator. Further, because of distortions of the earth's gravitational field in the equatorial plane, a resonance effect exists which can cause large deviations of the satellite from the desired position in the equatorial plane.[1]

The control process to be described is concerned with minimizing the errors in the three orbital parameters considered most significant. The most important error is in the length of semimajor axis a. An error in this parameter causes a secular drift of the satellite which, if left uncorrected, can carry it completely out of the antenna cones of view. Another significant orbital parameter error is the inclination of the orbital plane with respect to the equator. This error tends to grow due to the perturbing effects of the sun and moon. The average position of the satellite is also important if the periodic wandering motion is to stay entirely within the antenna cones of view. It is the sum of the squares of the errors in these three orbital parameters that the orbital control system attempts to minimize. The orbital control process is also designed to limit the amount of fuel consumed.

The complete guidance process, the mathematical relationships involved, and some details of simulating its operation on a digital computer are described in Ref. 2. This paper extends the work of Ref. 2 and discusses a different technique for the orbit transfer computation process. The orbit transfer process discussed here is formulated in terms of two-body orbital parameters and optimized by the techniques of dynamic programming. The orbit transfer process discussed in Ref. 2 is formulated in terms of inertial rectangular coordinates with no attempt to optimize.

In the discussion to follow, the equations of motion in terms of two-body orbital parameters are presented, the dynamic programming equations are given, and the integration of the orbital transfer process into the complete guidance loop is described. In Sec. VII, the numerical results for some typical cases are discussed, and some factors are pointed out which influence the design of a practical system.

II. Equations of Motion

For the 24-hr satellite the effects of perturbing forces will be small over the total time interval of the multistage orbit transfer process, and, consequently, the motion of the satellite may be assumed to follow a Keplerian ellipse during short periods of zero thrust. Such an ellipse is defined by a set of six orbital parameters. These parameters define the size and shape of the orbit, its orientation in inertial space, and the position of the satellite in the orbit with respect to time.

There are many possible sets of these orbital parameters. The set chosen here is particularly well suited for nearly circular orbits slightly inclined to the equator. These parameters and their use in astrodynamical computations are dis-

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cussed in Ref. 3. They are a, $e \cos E_0$, $e \sin E_0$, and the components of $\mathbf{U}_0$ and $\mathbf{V}_0$ where‡

a = semimajor axis
e = eccentricity
E_0 = value of eccentric anomaly at $t = t_0$
$\mathbf{U}_0$ = a unit vector directed from the center of force along the radius vector to the satellite at $t = t_0$
$\mathbf{V}_0$ = a unit vector 90° ahead of $\mathbf{U}_0$ in the orbital plane

The system state variables may be the six two-body orbital parameters defining the Keplerian two-body orbit along which the satellite moves while no thrust is applied. When a thrust vector is applied to the satellite, the state variables change continuously with time until the thrust is terminated. Thus, the state variable concept provides a convenient way of describing orbit transfer processes, since a set of six numerical values assigned to these parameters defines a particular Keplerian orbit.

When the difference between the orbital parameters for the initial and terminal orbits of the orbit transfer process is small, it is more convenient to define the state variables as the deviations of the actual orbital parameters from the desired terminal orbit parameters, i.e.,

$$\Delta p_0 = p_T - p_0$$

where Δp_0, p_T, and p_0 are 6×1 vectors. The components of p_T and p_0 are the six two-body orbital parameters for the terminal and initial orbits, respectively.

Instead of using three components of $\Delta\mathbf{U}_0$ and $\Delta\mathbf{V}_0$ as the two-body orbital parameters, the parameters $\Delta\tilde{u}_0$, $\Delta\tilde{v}_0$, and $\Delta\tilde{w}_0$ may be used.[3] These three parameters represent the small rotations of the actual orbit about the unit vectors $\mathbf{U}_0$, $\mathbf{V}_0$, and $\mathbf{W}$, which are required to rotate these unit vectors into coincidence with the corresponding unit vectors of the specified terminal orbit (Fig. 1). These small rotations are defined by[3]

$$\Delta\tilde{u}_0 = \mathbf{W}\cdot\Delta\mathbf{V}_0 = -\mathbf{V}_0\cdot\Delta\mathbf{W}$$

$$\Delta\tilde{v}_0 = \mathbf{U}_0\cdot\Delta\mathbf{W} = -\mathbf{W}\cdot\Delta\mathbf{U}_0$$

$$\Delta\tilde{w}_0 = \mathbf{V}_0\cdot\Delta\mathbf{U}_0 = -\mathbf{U}_0\cdot\Delta\mathbf{V}_0$$

where the unit vector $\mathbf{W}$ is defined by

$$\mathbf{W} = \mathbf{U}_0 \times \mathbf{V}_0$$

The state vector for the orbit transfer process may then be given by§

$$\Delta p = [\Delta a, \Delta(e \cos E_0), \Delta(e \sin E_0), \Delta\tilde{u}_0, \Delta\tilde{v}_0, \Delta\tilde{w}_0]^T$$

The orbit transfer process is described by considering the two-body orbital parameters as state variables. Therefore, it is necessary to obtain the equations of motion of the satellite in terms of these parameters as variables. The desired set of equations is obtained by starting with the equations of motion in terms of inertial rectangular coordinates and using the method of variation of parameters. The derivation for the set of orbital parameters just discussed is carried out in considerable detail in Ref. 4. The resulting set of equations is given in Fig. 2.

The equations of motion are nonlinear, and the orbital parameters enter into the matrices in a rather involved manner. However, for a multistage orbit transfer process, it may be assumed that the duration of each application of thrust is short relative to the orbital period and that the actual changes in the orbital parameters are small during any particular stage. When these conditions hold, the matrices may be considered constant during any one stage of the process.

‡ Only three of the six components of $\mathbf{U}_0$ and $\mathbf{V}_0$ need be known since these unit vectors satisfy $\mathbf{U}_0\cdot\mathbf{V}_0 = 0$, $\mathbf{U}_0\cdot\mathbf{U}_0 = 1 = \mathbf{V}_0\cdot\mathbf{V}_0$.

§ The superscript T denotes the transpose of a vector.

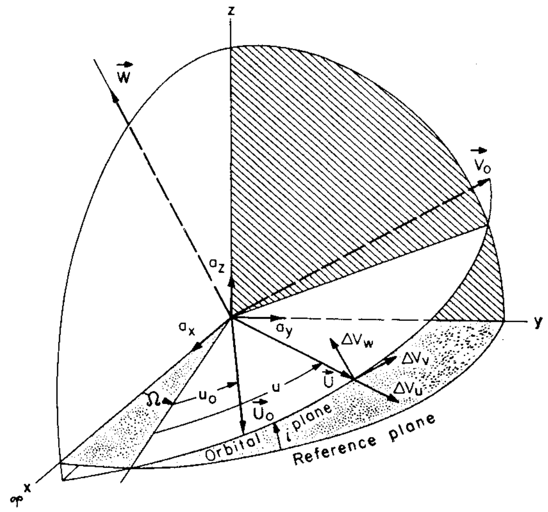

Fig. 1 Coordinate systems.

For the orbital transfer process considered in this paper, the system state vector is given by

$$\Delta p = [\Delta a, \Delta i, \Delta\tilde{w}_0]^T$$

The eccentricity is assumed zero in the equations in Fig. 2, and these equations are modified by neglecting $\Delta(e \cos E_0)$ and $\Delta(e \sin E_0)$ and replacing $\Delta\tilde{u}_0$ and $\Delta\tilde{v}_0$ by Δi. For the state variables Δa, Δi, and $\Delta\tilde{w}_0$, the resulting vector-matrix differential equation is given in Fig. 3.

III. Optimization of the Orbit Transfer Process

The process to be discussed involves transferring the satellite from some initial orbit, resulting from the injection cut-off conditions of the ascent phase, to a specified terminal orbit. This transfer process is assumed to consist of a sequence of discrete thrust vectors applied to the satellite. Each thrust vector transfers the satellite from one two-body orbit to another. Therefore, during the process the satellite moves from its initial two-body orbit through a series of intermediate two-body orbits until the specified

$$\begin{bmatrix} \frac{da}{d\tau} \\ \frac{d(e\operatorname{Cos}E_0)}{d\tau} \\ \frac{d(e\operatorname{Sin}E_0)}{d\tau} \\ \frac{d\tilde{u}_0}{d\tau} \\ \frac{d\tilde{v}_0}{d\tau} \\ \frac{d\tilde{w}_0}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2e\operatorname{Sin}E_0}{nr_0} & \frac{2e\operatorname{Cos}E_0}{nr_0} & 0 & 0 & \frac{2a\sqrt{1-e^2}}{nr_0^2} \\ \frac{2e\operatorname{Sin}E_0}{nr_0} & 0 & \frac{1-e^2}{nar_0} & 0 & 0 & \frac{2\sqrt{1-e^2}}{nar_0} \\ -\frac{2e\operatorname{Cos}E_0}{nr_0} & -\frac{1-e^2}{nar_0} & 0 & 0 & 0 & -\frac{\sqrt{1-e^2}\,e\operatorname{Sin}E_0}{nr_0^2} \\ 0 & 0 & 0 & 0 & -\frac{1}{na^2\sqrt{1-e^2}} & 0 \\ 0 & 0 & 0 & \frac{1}{na^2\sqrt{1-e^2}} & 0 & 0 \\ -\frac{2a\sqrt{1-e^2}}{nr_0^2} & -\frac{2\sqrt{1-e^2}}{nar_0} & \frac{\sqrt{1-e^2}\,e\operatorname{Sin}E_0}{nr_0^2} & 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} R_a & V_a & 0 \\ R_c & V_c & 0 \\ R_s & V_s & 0 \\ 0 & 0 & r\operatorname{Sin}(u-u_0) \\ 0 & 0 & -r\operatorname{Cos}(u-u_0) \\ 0 & r & 0 \end{bmatrix} \begin{bmatrix} f_u \\ f_v \\ f_w \end{bmatrix}$$

Fig. 2 Elliptic orbit equations.

$$\begin{bmatrix} \frac{da}{d\tau} \\ \frac{di}{d\tau} \\ \frac{d\tilde{w}_0}{d\tau} \end{bmatrix} = \frac{1}{na} \begin{bmatrix} 0 & 2a & 0 \\ 0 & 0 & \operatorname{Cos} u \\ -2\left[1-\operatorname{Cos}(u-u_0)\right] & \left[3(u-u_0)-4\operatorname{Sin}(u-u_0)\right] & 0 \end{bmatrix} \begin{bmatrix} f_u \\ f_v \\ f_w \end{bmatrix}$$

Fig. 3 Differential Equations in terms of a, i, and $\Delta\tilde{w}_0$.

terminal two-body orbit is reached. Each time interval of thrust application together with any time interval of zero thrust immediately preceding it is a stage of the process. Since the transfer process consists of a sequence of such stages, it is considered a multistage process.

Optimization of the general multistage orbit transfer process represents an N-dimensional minimization problem. A solution by classical minimization techniques requires solving N simultaneous equations. The use of dynamic programming can reduce the N-dimensional problem to N one-dimensional problems.[5] This makes the optimization problem much easier to solve. Further, when the system differential equations of motion are linear and the specified system performance index is quadratic, an analytic solution that includes certain types of constraints can be obtained.

The multistage process considered in this paper is optimized by choosing a sequence of vector increments to be added to the satellite's initial velocity vector in such a way that the specified system performance index is minimized. This process is carried out by first optimizing a one-stage process. Using Bellman's principle of optimality, a two-stage process is optimized and generalized to an N-stage process.[5, 6]

For brevity of notation, let the subscript k denote the conditions at the end of the kth stage of the process. The state of the system at the end of the $(k + 1)$th stage is then given by

$$p_{k+1} = p_k + A_k\Delta V_k$$

Subtracting p_T from both sides of the forementioned equation, multiplying through by -1, and noting that

$$\Delta p_j = p_T - p_j \qquad j = 0, \ldots, N-1$$

the state transformation equation is obtained for each stage of the process:

$$\Delta p_{k+1} = \Delta p_k - A_k\Delta V_k$$

This is the form of the state transformation equation to be used where matrix A_k is evaluated according to the state of the system at the end of the kth stage. The state transformation equation for the three state variables of the orbit transfer process under consideration is given in explicit form in Fig. 4. The three components of the incremental velocity vector ΔV are resolved along the unit vectors **U**, **V**, and **W** as indicated by the subscripts. The unit vector **U** is directed from the center of force to the satellite and rotates with it in the orbital plane. The unit vector **V** leads **U** by 90° in the orbital plane. The unit vector **W** is normal to the orbital plane (Fig. 1).

A performance index of a physical system is usually some measure of the deviation of the actual performance of the system from the desired or idealized performance The performance index generally includes the effects of one or more constraints on system behavior which prevent the idealized performance from being obtained. System performance is optimized by minimizing the deviation of the actual performance from the ideal performance. For the orbital transfer process under consideration, idealized performance constitutes driving the terminal errors in the three state variables to zero.

A constraint placed on the process requires that the sum of the squares of the magnitudes of the velocity vector increments during the transfer process be limited. This restriction is equivalent to limiting the mass of fuel consumed during the orbit transfer process.

The equation for the system performance index for an N-stage process based on the forementioned requirements is given by

$$J_N[\Delta p_0; \Delta V_0, \ldots, \Delta V_{N-1}] = \Delta p_N{}^T Q_N \Delta p_N + \lambda \sum_{k=0}^{N-1} \Delta V_k{}^T \Delta V_k$$

where the first term on the right represents the sum of the weighted squares of the orbital parameter errors at the termination of the process, and the second term represents the constraint on fuel consumption. The numerical value of the constant λ depends upon the upper bound assumed for fuel mass. The matrix Q_N weights the squares of the terminal errors in the three state variables, and Δp_N is given by the state transformation equation.

$$\Delta p_N = \Delta p_{N-1} - A_{N-1}\Delta V_{N-1}$$

where Δp_{N-1} is the state at the start of the final stage of the process.

System performance is optimized by choosing the set of ΔV vectors which minimizes the expression $J_N[\Delta p_0; \Delta V_0, \ldots, \Delta V_{N-1}]$. This choice is made by applying the technique of dynamic programming.

If the state transformation equation is linear and if the system performance index is quadratic, it is possible to derive analytically certain matrix recurrence relations that link the various stages together. Begin with some definitions. Let the function $f_N(\Delta p_0)$ be defined by the following: $f_N(\Delta p_0)$ is the cost of an orbit transfer process of N-stages duration, with the initial state Δp_0, and an optimum control policy being used. This function is the minimized performance index. An optimum control policy is the set of ΔV vectors minimizing the performance index J_N. By means of the principle of optimality, a recurrence relation can be derived from this definition of $f_N(\Delta p_0)$.

In terms of the orbit transfer problem, the principle of optimality may be stated as follows[6]: an optimal sequence of incremental velocity vectors $\Delta V_0, \Delta V_1, \ldots, \Delta V_{N-1}$ has the property that whatever the initial state Δp_0 may be and whatever choice is made for ΔV_0, the remaining sequence $\Delta V_1, \ldots, \Delta V_{N-1}$ must constitute an optimal sequence with regard to the state Δp_1 resulting from the choice of ΔV_0. One is initially confronted with an N-stage process starting from state Δp_0. The choice of ΔV_0 transforms the system to some state Δp_1 and an $(N - 1)$-stage process remains. The minimized performance index for the $(N - 1)$th stage starting from state Δp_1 is $f_{N-1}(\Delta p_1)$ or, by the state trans-

$$\begin{bmatrix} \Delta a \\ \Delta i \\ \Delta\tilde{w}_0 \end{bmatrix}_k = \begin{bmatrix} \Delta a \\ \Delta i \\ \Delta\tilde{w}_0 \end{bmatrix}_{k-1} - \frac{1}{n_{k-1}a_{k-1}} \begin{bmatrix} 0 & 2a_{k-1} & 0 \\ 0 & 0 & \operatorname{Cos} u_{k-1} \\ -2\left[1-\operatorname{Cos}(u-u_0)_{k-1}\right] & \left[3(u-u_0)_{k-1}-4\operatorname{Sin}(u-u_0)_{k-1}\right] & 0 \end{bmatrix} \begin{bmatrix} \Delta V_u \\ \Delta V_v \\ \Delta V_w \end{bmatrix}_{k-1}$$

$$\Delta p_k = \Delta p_{k-1} - A_{k-1}\Delta V_{k-1}$$

Fig. 4 State transformation equation for Δa, Δi, and Δw_0.

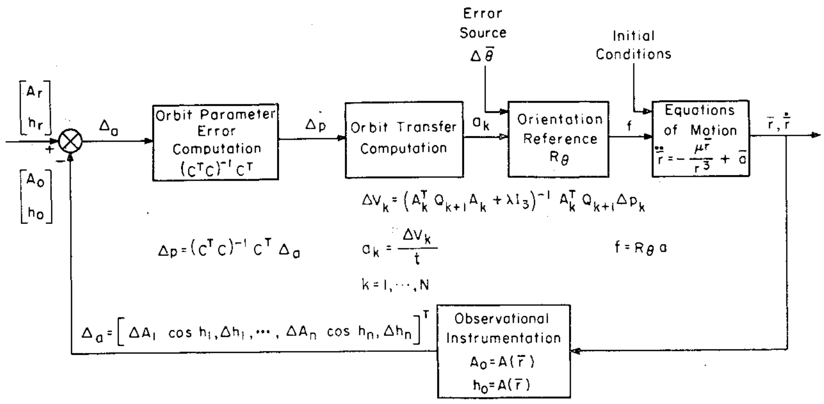

Fig. 5 System functional block diagram.

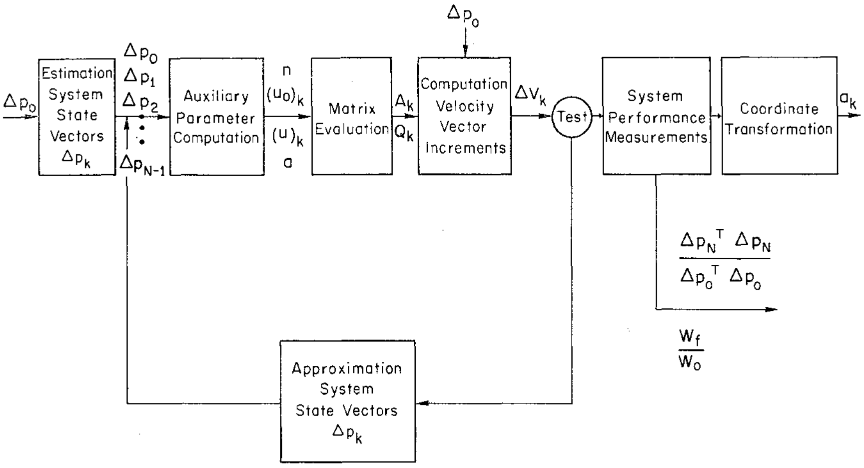

Fig. 6 Functional block diagram for orbit transfer optimization process.

formation equation, f_{N-1} $(\Delta p_0 - A_0\Delta V_0)$. Thus, the recurrence relation for the N-stage process is

$$f_N(\Delta p_0) = \min_{\Delta V_0} \{\lambda \Delta V_0{}^T \Delta V_0 + f_{N-1}(\Delta p_0 - A_0\, \Delta V_0)\}$$
$$f_1(\Delta p_0) = \Delta p_0{}^T\, Q_0\, \Delta p_0 \qquad N = 2, 3, \ldots$$

In the explicit form this recurrence relation becomes a set of matrix recurrence equations. These equations, which are derived in detail in Refs. 6 and 7, are given by‖

$$Q_{N-r} = Q_{N-r+1} - Q^T{}_{N-r+1}A_{N-r}[(A^T{}_{N-r}\, Q_{N-r+1}A_{N-r} + \lambda I_3)^{-1}]^T\, A^T{}_{N-r}\, Q_{N-r+1}$$

$$\Delta V_{N-r} = (A^T{}_{N-r}\, Q_{N-r+1}\, A_{N-r} + \lambda I_3)^{-1}\, A^T{}_{N-r} \times Q_{N-r+1}\, \Delta p_{N-r}$$

$$\Delta p_{N-r} = \Delta p_{N-r-1} - A_{N-r-1}\, \Delta V_{N-r-1}$$

$$\min J_N[\Delta p_0;\, \Delta V_0, \ldots, \Delta V_{N-1}] = \Delta p_0{}^T Q_0\, \Delta p_0$$

where $r = 1, \ldots, N$.

It may be observed that the minimum value of J_N depends only upon the initial state Δp_0 and the number of stages through the matrix Q_0. The A matrix changes from stage to stage as the system state and the time vary.

IV. Integrating the Optimum Orbit Transfer Process with the System Control Loop

A functional block diagram for the control loop operation is given in Fig. 5. The "orbit transfer computation" block shown in the diagram represents the computation of corrective accelerations, in accordance with the optimization process described in Sec. III.

Since the function of the guidance process is to match closely both the position and velocity of a satellite with a hypothetical point (or satellite) moving in the reference orbit, the system must first determine some measure of the position and velocity errors of the actual satellite with respect to the reference satellite. The correction of the orbit transfer process requires a knowledge of these errors in the form of six orbital parameter errors Δp. Therefore, the first step in the guidance process is to transform the desired error measurements to this form. The input for the system consists of $2m$ $(m \geq 3)$ observation residuals formed from m ground observations of azimuth and elevation angles. In direct contrast to conventional astronomical usage, these residuals have been defined to correspond to correction of the observed values to match the reference values. Theoretically, only six observation residuals are required for the determination of the six orbital parameter errors. However, greater accuracy may be achieved by taking more than six residuals and using least squares techniques. The basic relationship between the residuals and parameter errors may be expressed in vector-matrix form as shown in Fig. 5.

The inputs to the "orbit transfer computation" are the six orbital parameter corrections plus six constants needed in the dynamic programming routine. This routine is an iterative process for calculating the optimal set of incremental velocity vectors for transferring the satellite to the reference point. These corrective velocity vectors, which constitute the input to the "orientation reference" block in Fig. 5, are to be applied to the satellite for 1-min durations at equally spaced time intervals. The "orientation reference" or attitude control system determines the direction of each thrust. The application of these corrective velocity increments to the satellite completes the orbit transfer process. In a real case the required orbit corrections are determined by taking a set of ground observations. The determination of these corrections in the simulated problem, however, requires a precise solution of the equations of motion. The output of the "equations of motion" block shown in Fig. 5 will be in the form of position and velocity components of the satellite with respect to the inertial coordinate system. These are then transformed into azimuth and elevation angles from which the residuals and the orbital parameter corrections are computed.

‖ These equations apply to both the three-dimensional and six-dimensional cases.

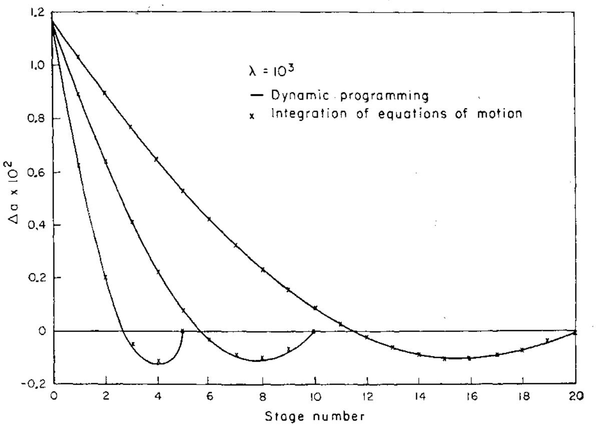

Fig. 7 Change in Δa during orbit transfer process.

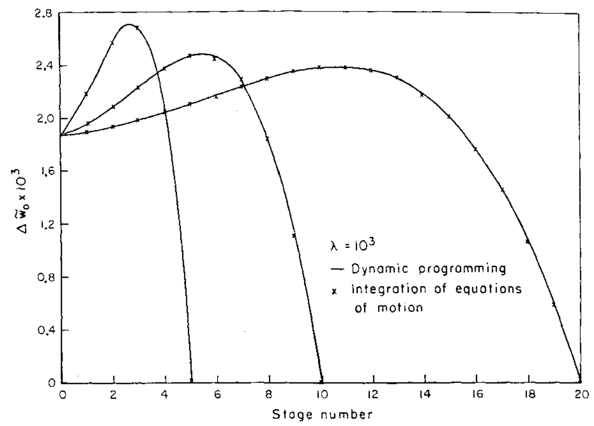

Fig. 8 Change in $\Delta\tilde{w}_0$. during orbit transfer process.

V. Computational Procedure for Orbit Transfer Optimization Process

A functional block diagram for the N-stage orbit transfer optimization process is given in Fig. 6. This diagram represents the "orbit transfer computation" block in Fig. 5. The input to this process is the system state vector Δp_0 determined by the differential correction process based on observations of the actual satellite position. In order to use the dynamic programming routine, however, one must also have information about the other $N - 1$ intermediate states of the system during the N-stage transfer. Since the nature of this problem requires the use of dynamic programming in an iterative process, a rough estimate of these state vectors suffices to start the procedure.

From the estimated Δp vectors, the auxiliary parameters u, u_0, a, and n may be computed for each state, giving sufficient information for evaluating the A matrices. These are then used in the Q matrix recurrence relation in Sec. III to determine the set of Q matrices. When the two sets of matrices have been evaluated, the velocity vector increments may be computed by alternately solving the equations for ΔV and Δp for each stage in the process. For the numerical problem to be considered, the recurrence relations are three-dimensional so that the computed Δp vectors are 3×1 vectors of the form

$$\Delta p = [\Delta a, \Delta i, \Delta\tilde{w}_0]^T$$

However, it is also necessary to compute the corresponding 6×1 Δp vector for each stage in the form

$$\Delta p = [\Delta a, \Delta(e \cos E_0), \Delta(e \sin E_0), \Delta\tilde{u}_0, \Delta\tilde{v}_0, \Delta\tilde{w}_0]^T$$

since five of these original parameter errors are used in the auxiliary parameter computation. Therefore, once a set of ΔV vectors is obtained, the next iteration may be prepared for by evaluating the set of 6×3 matrices (Fig. 2) to be used with the incremental velocity vectors in the state transformation equation#

$$\Delta p_{k+1} = \Delta p_k - A_k \, \Delta V_k \qquad k = 0, \ldots, N-2$$

in order to determine the new set of 6×1 system state vectors. These latter values replace the original estimates and become the initial conditions for the next iteration. The optimal set of incremental velocity vectors may be found by repeating the process until the difference from one iteration to the next is negligible.

The "coordinate transformation" block shown in Fig. 6 involves the transformation of the N incremental velocity vectors whose components are referred to the **U**, **V**, **W** coordinate system (Fig. 1) to N acceleration vectors whose components are referred to the inertial coordinate system. As shown in Fig. 6, this "coordinate transformation" completes the "orbit transfer computation" and provides the input to the "orientation reference" block (Fig. 5).

VI. Discussion

The dynamic programming computational procedure was tested for convergence by comparing the system state vectors at each stage for successive iterations. The numerical results indicate that the parameter errors Δa and $\Delta\tilde{w}_0$ at each stage of the process converge by the second iteration. More specifically, the values of these variables at each stage for the second and third iterations agree to five significant figures or to 10^{-8}. The parameter error Δi, on the other hand, does not converge with succeeding iterations but, rather, alternately approaches two different sets of values. This is due to a computational difficulty for orbits with small inclinations, since for these orbits any slight correction in Δi may cause a drastic shift in the position of the node. The amount of this shift is impossible to predict with any accuracy since the function is so poorly defined for small changes in Δi. As a result, one cannot predict the values of the angles u_0 (and u) for each stage. This causes a difficulty in computing the set of values for ΔV_w and consequently for Δi.**

In addition to the convergence test, an investigation was made to determine the effect of the linearization assumptions on the accuracy of the dynamic programming solution. This was done by comparing the final set of system state vectors computed by the dynamic programming routine with the true values determined by the precise numerical integration of the two-body equations of motion. Figures 7 and 8 show that for the parameter errors Δa and $\Delta\tilde{w}_0$, the dynamic programming solution agrees with the actual satellite solution to three significant figures or to 10^{-5} and 10^{-6}, respectively. Figure 9, on the other hand, shows that the

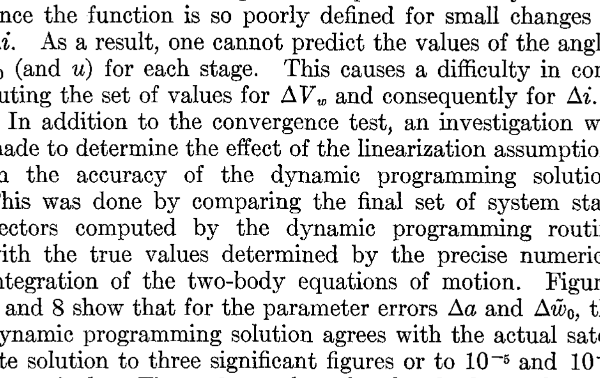

Fig. 9 Change in Δi during orbit transfer process.

Eccentricity is set equal to zero in the matrices in Fig. 2.

** By using the equations for $\Delta\tilde{u}_0$ and $\Delta\tilde{v}_0$ from Fig. 2 instead of the equation for Δi, this computational problem could be eliminated.

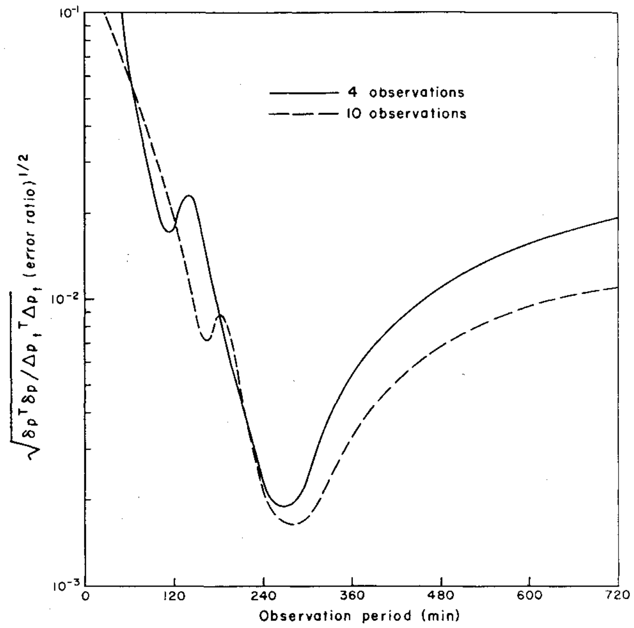

Fig. 10 Effect of observation period on differential correction process error ratio.

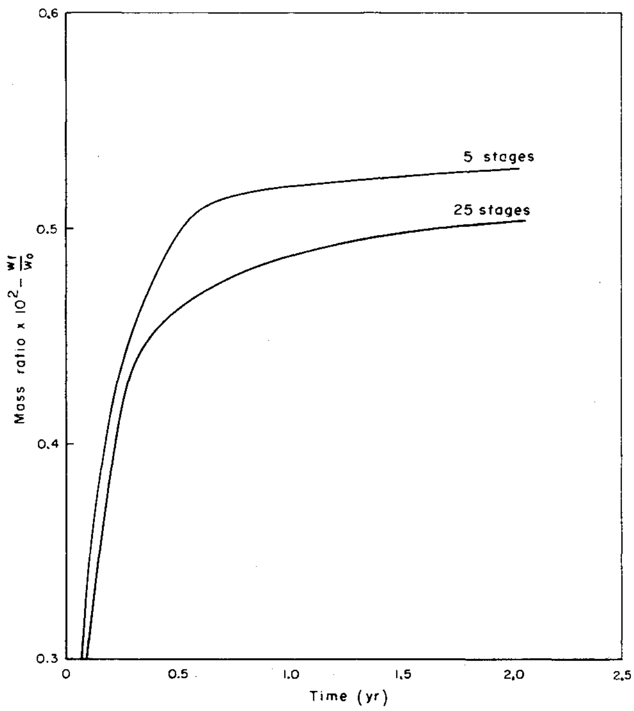

Fig. 11 Mass ratio vs time for satellite to drift 5° from desired longitude.

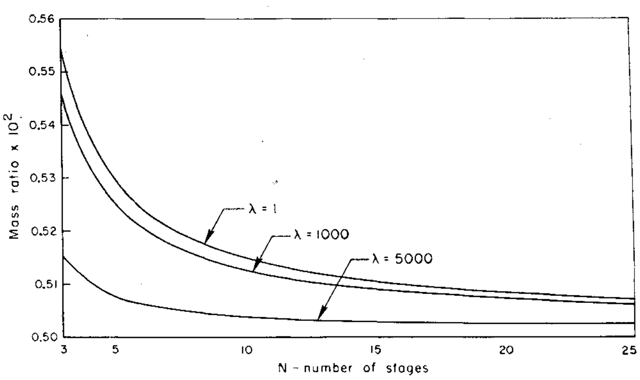

Fig. 12 Mass ratio vs number of stages in the process.

dynamic programming and satellite solutions for Δi agree to a point and then begin to diverge. It can be shown, however, that this divergence is due to a computational problem concerning Δi and is not a result of the linearization assumptions.

The satellite orbit resulting from injection conditions is determined by a differential correction process.[8] The azimuth and elevation angle observation residuals are expressed as linear combinations of the deviations of the actual orbital parameters from the corresponding parameters of the reference orbit. The coefficients of the orbital parameter deviations (or corrections) are the first-order partial derivatives from Taylor's series expansions of the azimuth and elevation angles in terms of small changes in the orbital parameters of the reference orbit. These derivatives are numerically evaluated for the orbital parameters of the reference orbit and the observation time associated with the particular observation residual represented. Because of the omission of higher-order derivatives, the size of the observation residuals that this correction process can handle is limited.

The curves in Fig. 10 indicate that there is an optimum time interval in which to make the observations. The existence of this optimum observation period arises from two separate effects:

1) Computational round-off and observational errors become more significant for short observation periods.

2) Drift of the satellite away from the desired reference position causes the observation residuals to become too large for the useful linear range of the process for long observation periods.

Figure 10 indicates an optimum observing period of approximately 270 min for the initial orbit used to obtain the data. A special initial orbit with zero error in the semimajor axis (i.e., no drift) indicated that the error ratio did not exhibit a minimum but continued to decrease slowly with an increasing observation period.

Figure 11 gives a plot of mass ratio against time in years to drift 5° in longitude. The mass ratio is defined by

$$\frac{W_f}{W_0} = \frac{\text{mass of fuel consumed}}{\text{initial mass of vehicle plus fuel}}$$

$$= 1 - \exp\left(- \frac{1}{gI} \sum_{k=0}^{N-1} |\Delta V_k| \right)$$

where

gI = 0.009344679 earth radii/min

$|\Delta V_k|$ = magnitude of velocity increment acquired during $(k+1)$th stage

The curves in Fig. 11 were computed as follows: by letting the Lagrange multiplier λ increase in the system performance index, the sum of the squares of the magnitudes of the ve-

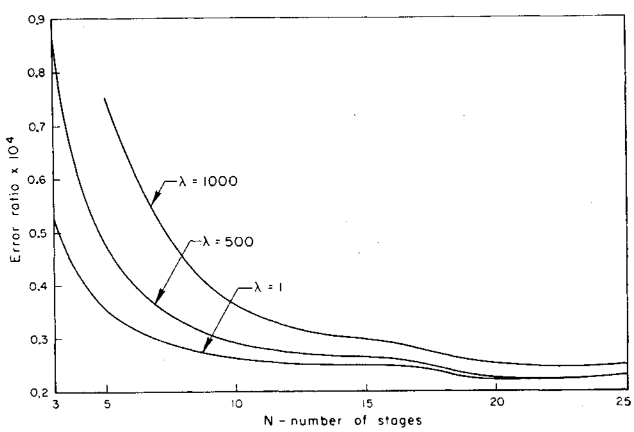

Fig. 13 Error ratio vs number of stages in the process.

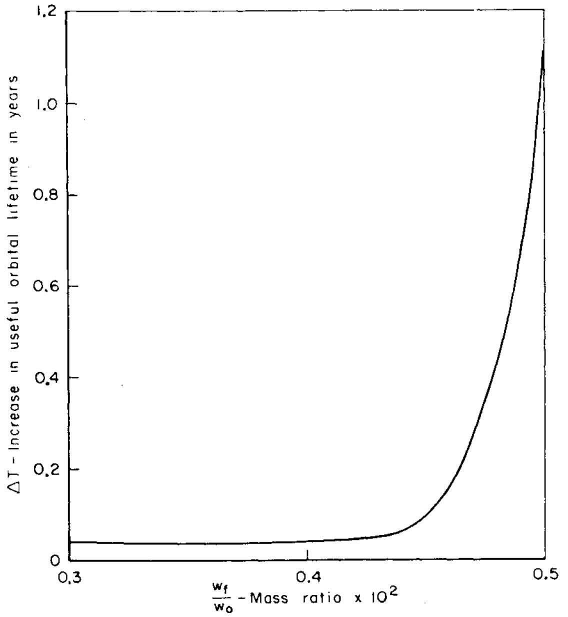

Fig. 14 Increase in useful orbital lifetime of a 25-stage process over a five-stage process.

locity vector increments is weighted more and more heavily in the minimization process at the expense of increased terminal errors. This, in essence, means that as λ is increased, less fuel is used during the orbit transfer process, but the terminal errors are larger. The larger the terminal errors, the larger the secular drift due to the error in the semimajor axis, and the shorter the duration of time to drift through 5° of longitude. The two curves in Fig. 11 do not include the effect of differential correction process errors. These errors do not significantly affect the mass ratios but do greatly increase the drift rate. Thus, the principal effect of the differential correction errors will be to compress the curves in a direction parallel to the time axis. It should be pointed out that correcting only Δa and $\Delta \tilde{w}_0$ by the orbital control process consumes less fuel than attempting to correct the initial errors in all six orbital parameters.

VII. Practical Considerations Concerning System Design

An obvious problem in the design of a hardware system to mechanize the orbit transfer process is how many stages to use. The data in Figs. 12 and 13 indicate that both the mass ratio and the error ratio decrease as the number of stages increases for a given value of λ.

The data in Fig. 11 have been replotted in Fig. 14 to show the increase in useful orbital lifetime for a specified available mass ratio by increasing the number of stages in the process from 5 to 25. Useful orbital lifetime is defined as the time in years to drift 5° in longitude from the desired point over the equator. Figure 14 indicates that a 25-stage process has little advantage over a five-stage process for small mass ratios. However, permitting the mass ratio to increase from 0.004 to 0.005 gives a much larger increase in the useful orbital lifetime with a 25-stage process than with a five-stage process. It should be remembered that any point on this curve is determined by adjusting the value of λ in the computation of the set of optimum velocity vector increments applied to the satellite to accomplish the orbit transfer. It should also be remembered that Fig. 14 does not include differential correction process errors. These errors will tend to reduce the useful orbital lifetime.

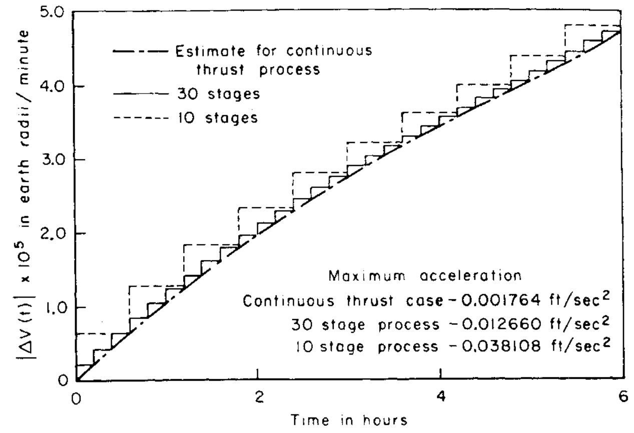

Fig. 15 Magnitude of velocity vector increment as a function of time.

The data in Figs. 12 and 13 indicate little change in mass ratio, error ratio, and the manner in which the state variables behave during the orbit transfer process when the number of stages exceeds 20. This suggests that the performance of a process with 20 or more stages begins to approximate closely the performance of a continuous process. This immediately raises a question concerning the possible superiority of a continuous process over a discrete process. The behavior of the magnitude of the velocity vector increment with time is shown in Fig. 15. The vertical segments of the staircase curves for the 10- and 30-stage processes actually have a finite slope. The period of thrust during each stage is assumed to exist for 1 min.

An estimate of the time behavior of the magnitude of vector ΔV is indicated by the dot-dash curve in Fig. 15 for the case where the number of stages becomes infinite, i.e., the continuous process. The maximum acceleration for each case is indicated on the figure. For the continuous thrust curve it is the slope of its curve through the origin. For the discontinuous cases it is the maximum velocity increment divided by 1 min. Clearly, the continuous thrust case has a big advantage in regard to the acceleration that the satellite is subjected to during the orbit transfer process. From the forementioned discussion it appears that the continuous process offers an improvement over a discrete process from the standpoint of accuracy, fuel economy, and lower acceleration during the orbit transfer.

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